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Aspects of multistation quantum information broadcasting

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Abstract

We study quantum information transmission over multiparty quantum channel. In particular, we show an equivalence of different capacity notions and provide a multiletter characterization of a capacity region for a general quantum channel with k senders and m receivers. We point out natural generalizations to the case of two-way classical communication capacity.

Keywords: quantum information transmission, quantum broadcast channel, capacity theorem, capacity region, fidelity
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1. Introduction

Quantum channels have been in the field of interest since the early stages of the development of quantum information theory. However, the major progress in the domain have been achieved in the case of quantum channels with single both sender and receiver, so–called bipartite or single user channels [1, 2, 3, 4, 5, 6, 7]. Thorough investigations resulted in the quantum coding theorem which was conjectured to exist in the form analogous to the form of Shannon's theory [8, 9, 10, 11]. Various aspects of single user communication with assistance of different kind have been deeply analyzed (see [12] for the hierarchic classification of capacities in such scenarios). Nonadditivity of quantum channel capacity has been also reported [13]. Recently some progress has been achieved in the case of multiuser communication scenarios with both new aspects and some generalizations of known results considered [14, 15, 16].

In the paper we consider quantum information transmission over quantum channels. For a recent development in classical or secret information capacities see *e.g.* [17, 18, 19].

The paper deals with a multiparty communication. First we systemize the notions of quantum channel capacity in this setup. These are the generalizations of the ones from the single user channel's theory [20] and are known as entanglement transmission, subspace transmission and entanglement generation. We follow with the demonstration of the equivalence of this scenarios. In the second part we provide a capacity theorem

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with a simple proof for class of multiple antenna quantum broadcast channels. Further we point out natural generalization to the case of quantum capacity of a quantum channel assisted by two-way classical side channel. Finally we summarize and discuss our results.

2. Background

In this section we provide a short introduction into the area of quantum information transmission and a detailed background for further considerations.

A notion of a quantum channel introduced below is a standard mathematical notion used for the description of a physical disturbance to the quantum systems caused by the unavoidable interaction with an environment. Our main concern will be a quantitative description of the issue of quantum information transmission through such channel. Throughout the paper a shorthand notation $\psi \equiv |\psi\rangle\langle\psi|$, logarithms are taken to base 2.

2.1. General view on the communication over quantum channels

By the definition multiparty quantum channel with k inputs and m outputs (km-user channel, in short km-UC) is a completely positive trace preserving (CPTP) linear map Λ acting from input density operators in $\mathcal{B}(\mathcal{H}_{iCh}) = \bigotimes_{j=1}^{k} \mathcal{B}(\mathcal{H}_{iCh_j})$ to output density operators in $\mathcal{B}(\mathcal{H}_{oCh}) = \bigotimes_{i=1}^{m} \mathcal{B}(\mathcal{H}_{oCh_i})$, which in general can be of different dimensions. With this denotation it can be formally written as $\Lambda : \mathcal{B}(\mathcal{H}_{iCh}) \to \mathcal{B}(\mathcal{H}_{oCh})$.

We will consider a situation in which spatially separated k parties, denoted $\mathbf{A} = \{\mathbf{A}_i\}_{i\in\mathcal{K}=\{1,2,\ldots,k\}}$ and called Alicias, wish to communicate in a quantum manner spatially separated m parties, denoted $\mathbf{B} = \{\mathbf{B}_i\}_{i\in\mathcal{M}=\{1,2,\ldots,m\}}$ and called Bobbys. Quantum information embodied in quantum systems sent by members of $\mathbf{A}$ is physically altered what is described by km-UC. An implicit assumption of both classical and quantum information theory is that Alicias and Bobbys have at their dispose n ($n \to \infty$) instances of such channel which is usually written as $\Lambda^{\otimes n}$ (this contains the assumption that the channel is memoryless). We assume that both groups act cooperatively *i.e.* they agree to follow some jointly determined protocol which goal is, using $\Lambda^{\otimes n}$, to establish a nontrivial reliable quantum communication channel between specified nodes of the network capable of faithful quantum information exchange. We will use single indices from the set G to specify all two–nodes connections in the network. Obviously the number of elements in G is km, however we will use $|G|$ for this number as this will allow for more clarity. The set is further divided into subsets on senders' and receivers' side *i.e.* $G = \{G^{(j)}\}_j$ (note that the division differs for the parties on both ends of the channel).

Due to the different goals $\mathbf{A}$ and $\mathbf{B}$ want to achieve we have different definitions of capacities *i.e.* different approaches to the problem of information transmission which we review below.

2.2. Review of quantum communication notions

2.2.1. Entanglement transmission

We start our review with a concept of entanglement transmission [1, 3].

We define the quantum sources $\mathfrak{S}_i = (\varrho_{A_i}^{(n)}, \mathcal{H}_{A_i}^{(n)})_{n\in\mathbb{N}}$, $i \in I$, to be the pairs of sequences of Hilbert spaces and block density matrices on them [4]. To the sources we assign entropy rates which are defined through $R_S(\mathfrak{S}_i) \equiv \limsup_{n\to\infty} S(\varrho^{(n)})/n := \mathfrak{R}_i^e$; S stands for

von Neumann entropy, I is some set of indices. It is assumed that every $\varrho_{A_i}^{(n)}$ is the part of the larger system $(RA)_i$ in some pure entangled state, *i.e.* $\varrho_{A_i}^{(n)} \equiv \mathrm{tr}_{R_i} \Psi_{(RA)_i}^{(n)}$ with the purifying system $\mathbf{R}$ assumed to be out of control of the parties. Note that we can always look at the density matrix in this way.

The following sequences of operations constitute the protocol: (i) Alicias' CPTP collective encodings $\mathcal{E}^{(n)} = \otimes_{j\in\mathcal{K}} \mathcal{E}_j^{(n)}$, $\mathcal{E}_j^{(n)} : \mathcal{B}(\mathcal{H}_{\mathbf{A}_j}^{(n)}) \to \mathcal{B}(\mathcal{H}_{iCh_j}^{\otimes n})$, $\mathcal{H}_{\mathbf{A}_j}^{(n)} = \bigotimes_{i\in G^{(j)}} \mathcal{H}_{A_i}^{(n)}$, (ii) Bobbys' collectively CPTP decodings $\mathcal{D}^{(n)} = \otimes_{j\in\mathcal{M}} \mathcal{D}_j^{(n)}$, $\mathcal{D}_j^{(n)} : \mathcal{B}(\mathcal{H}_{oCh_j}^{\otimes n}) \to \mathcal{B}(\mathcal{H}_{\mathbf{B}_j}^{(n)})$, $\mathcal{H}_{\mathbf{B}_j}^{(n)} = \bigotimes_{i\in G^{(j)}} \mathcal{H}_{B_i}^{(n)}$. The protocol together with the in-between usage of the channel $\Lambda^{\otimes n}$ results in a sequence of channels $\mathcal{N}_{\mathbf{A}\to\mathbf{B}}^{(n)}$.

One says that the sources $\mathfrak{S}_i$, $i \in G$, can be sent successfully (reliably) if there exists a protocol for which entanglement fidelity defined as

$$F_e\left(\bigotimes_{i\in G} \varrho_{A_i}^{(n)}, \mathcal{N}_{\mathbf{A}\to\mathbf{B}}^{(n)}\right) \equiv \mathrm{tr}\left[\mathcal{I}^{\mathbf{R}} \otimes \mathcal{N}_{\mathbf{A}\to\mathbf{B}}^{(n)}\left(\bigotimes_{i\in G} \Psi_{(RA)_i}^{(n)}\right) \bigotimes_{i\in G} \Psi_{(RA)_i}^{(n)}\right]$$

tends to one in the limit of large n. The $|G|$–tuple of rates $\{\mathfrak{R}_i^e\}_{i\in G}$ is said to be achievable if there exist sources with rates $\mathfrak{R}_i^e$ that can be sent reliably. *Quantum channel capacity is defined to be a closure of the set of all km-tuples of achievable rates.* The entanglement transmission capacity region will be denoted by $\mathcal{Q}_e$. To prevent unreasonable situations in which rates are infinite we concentrate only on sources satisfying quantum asymptotic equipartition property (QAEP; see [20]).

If we take input states to be maximally entangled we arrive at the notion of maximal entanglement transmission. The measure of reliability is called channel fidelity and is denoted by F_c. A symbol $\mathcal{Q}_m$ will be used for the capacity region.

The fidelity used above is called global. As shown in [16] global fidelity is equivalent to so called local ones (*i.e.* convergence in global fidelity implies convergence in all local fidelities and *vice versa*) which are defined by ($i \in G$)

$$F_e^{(i)}\left(\bigotimes_{i\in G} \varrho_{A_i}^{(n)}, \mathcal{N}_{\mathbf{A}\to\mathbf{B}}^{(n)}\right) \equiv \mathrm{tr}\left[\left[\mathrm{tr}_{\mathbf{RB}\setminus(RB)_i} \mathcal{I}^{\mathbf{R}} \otimes \mathcal{N}_{\mathbf{A}\to\mathbf{B}}^{(n)}\left(\bigotimes_{i\in G} \Psi_{(RA)_i}^{(n)}\right)\right] \Psi_{(RA)_i}^{(n)}\right],$$

where the partial trance means we trace out all the systems except $(RB)_i$. We adopt the convention in which one of the arguments of the fidelity is not the purification of ϱ but only ϱ itself. This is because fidelities do not depend upon specific purification. In the paper we also make use of group fidelities, which are the ones with the specified significant users traced out (obviously they are equivalent to local and global fidelities). These are denoted by $F^{[\mathcal{G}]}$ with $\mathcal{G}$ being any subset of G. We often omit one or both arguments of fidelities and freely write $F(\Lambda)$ or F with proper superscripts causing no confusion as the arguments are clear from the context. Absence of superscripts means we are considering global fidelities. This also concerns other fidelities considered further.

The definition of capacity region is general and is the same in all notions of capacity.

2.2.2. Subspace transmission

In the scenario of subspace transmission [5] Alicias and Bobbys wish to transmit arbitrary pure states drawn from some Hilbert spaces. One says that the sequence of Hilbert

spaces $\mathcal{H}_{A_i}^{(n)}$, $i \in G$, can be transmitted reliably if Alicias and Bobbys can use the protocol in such a manner that minimum pure state fidelity defined as

$$F_s\left(\bigotimes_{i\in G}\mathcal{H}_{A_i}^{(n)},\mathcal{N}_{\mathbf{A}\to\mathbf{B}}^{(n)}\right) \equiv \min_{\bigotimes_{i\in G}|\psi_{A_i}^{(n)}\rangle\in\bigotimes_{i\in G}\mathcal{H}_{A_i}^{(n)}} \mathrm{tr}\left[\mathcal{N}_{\mathbf{A}\to\mathbf{B}}^{(n)}\left(\bigotimes_{i\in G}\psi_{A_i}^{(n)}\right)\bigotimes_{i\in G}\psi_{A_i}^{(n)}\right] \qquad (1)$$

tends to one in the limit of large n. The $|G|$–tuple of rates $\{\mathfrak{R}_i^s\}_{i\in G}$ is said to be achievable if there exist sequences of Hilbert spaces $\mathcal{H}_{A_i}^{(n)}$, $i \in G$, with $\limsup_{n\to\infty}(\log\dim\mathcal{H}_{A_i}^{(n)}/n) = \mathfrak{R}_i^s$ which can be sent reliably. Capacity region is here denoted by $\mathcal{Q}_s$.

A similar scenario arises when we choose average fidelity as the reliability measure, *i.e.*

$$\bar{F}_s\left(\bigotimes_{i\in G}\mathcal{H}_{A_i}^{(n)},\mathcal{N}_{\mathbf{A}\to\mathbf{B}}^{(n)}\right) = \int \Pi_{i\in G}\mathrm{d}|\psi_{A_i}^{(n)}\rangle \mathrm{tr}\left[\mathcal{N}_{\mathbf{A}\to\mathbf{B}}^{(n)}\left(\bigotimes_{i\in G}\psi_{A_i}^{(n)}\right)\bigotimes_{i\in G}\psi_{A_i}^{(n)}\right],$$

where the integral is to be understood as $\int \mathrm{d}|\psi\rangle f(|\psi\rangle) = \int \mathrm{d}U f(U|\psi_0\rangle)$ with arbitrary $|\psi_0\rangle$ and RHS integral over all unitaries chosen according to the Haar measure on the subspace of interest. Quantities which are averaged are called pure state fidelities (pure state fidelity for a state φ and the channel will be denoted $F_s(\varphi,\Lambda)$). Rates are defined as above and we use a denotation $\bar{\mathcal{Q}}_s$ for the capacity region.

2.2.3. Entanglement generation

Last considered here is the entanglement generation introduced in [11]. The goal is to produce maximally entangled states between parties, $i \in G$. The first step of the protocol is replaced now by the preparation of a pure state $\bigotimes_{j=1}^k \Psi_{(\mathbf{AA}')_j}^{(n)}$, $\Psi_{(\mathbf{AA}')_l}^{(n)} \in \mathcal{H}_{\mathbf{A_l}} \otimes \mathcal{H}_{iCh_l}^{\otimes n}$ (there is no further preprocessing) as the input to the channel. The sequence arising from the protocol and the channel $\mathcal{N}_{\mathbf{A}'\to\mathbf{B}}^{(n)}$ is the concatenation of only the action of a channel and the decodings. Generation of some fixed $|\Phi_{d_i^{(n)}(AB)_i}^{(+)}\rangle = 1/\sqrt{d_i^{(n)}}\sum_{\gamma=0}^{d_i^{(n)}-1}|\gamma_{A_i}\rangle|\gamma_{B_i}\rangle$ with a given protocol is said to be reliable if entanglement generation fidelity defined as

$$F_g\left(\bigotimes_{i\in G}\Phi_{d_i^{(n)}(AB)_i}^{(+)},\mathcal{N}_{\mathbf{A}'\to\mathbf{B}}^{(n)}\right) \equiv \mathrm{tr}\left[\mathcal{I}^{\mathbf{A}}\otimes\mathcal{N}_{\mathbf{A}'\to\mathbf{B}}^{(n)}\left(\bigotimes_{j=1}^k\Psi_{(\mathbf{AA}')_j}^{(n)}\right)\bigotimes_{i\in G}\Phi_{d_i^{(n)}(AB)_i}^{(+)}\right],$$

tends to one in the limit of large n. One says that the $|G|$–tuple of rates $\{\mathfrak{R}_i^g\}_{i\in G}$ is achievable if there is a sequence of preparations allowing for reliable generation of maximally entangled states with $\limsup_{n\to\infty}(\log d_i^{(n)}/n) = \mathfrak{R}_i^g$. Capacity region is defined in analogy to the previous scenarios and is denoted by $\mathcal{Q}_g$.

There is no need to permit Alicias perform encodings as this would only mean that we let them prepare mixed instead of pure states at the beginning of the protocol, which does not provide us with substantially different communication scenario (cf. Section 3). However, when classical support comes into play (see the next subsection) it is reasonable to consider Alicias' operations (preprocessing as well as operations during execution of the protocol). To reduce the clutter we use the same denotation for both scenarios.

2.3. *Classical communication as a supportive resource*

So far we have not mentioned anything about additional resources which may be used to enhance quantum transmission. Usually we let the parties share entanglement, randomness, classical secret bits or communicate classically (without any cost). In this paper we will be mainly concerned with a special case of the last possibility, namely one-way forward classical support denoted with a superscript $\rightarrow$, *e.g.* $\mathcal{Q}_s^{\rightarrow}$. It is instructive to realize how the classical support fits into the quantum operation approach. The connection is made by generalized measurements performed by Alicias. Learning upon the classical results i (i is a multiindex) of such measurements Alicias choose to perform $\mathcal{E}_i$ (which are trace-decreasing, *i.e.* probabilistic, quantum operations) and inform Bobbys about the value of i who can perform appropriate $\mathcal{D}_i$ (which are trace preserving, *i.e.* deterministic operations). It is now clear that entanglement generation in this scenario makes sense only if senders are allowed to operate on their parts, which was not the case in a zero-way regime. In a similar fashion we construct one-way backward and two-way protocols. In case of single user channels there is a well known result stating uselessness of one–way forward classical support [2, 4]. Recently the result has been generalized [16].

2.4. *Coherent information*

Here we recall one more quantity great importance of which was conjectured long before its full recognition. It is the coherent information [3, 8], playing a role similar to that of the mutual information in classical information theory, defined as $I_c(X > Y)_{\varrho^{AB}}$; $X = A, B$; $Y = B, A$. We are not going into details concerning similarities and differences between coherent and mutual information (for a recent result see [22]). We recall only one important feature, namely quantum data processing inequality which states that coherent information never increases in state postprocessing (operations $\mathcal{D}_{B\rightarrow B'}$ on B side), *i.e.* $I_c(A > B)_{\varrho^{AB}} \geq I_c(A > B')_{\mathcal{D}_{B\rightarrow B'}(\varrho^{AB})}$ [3].

3. Equivalence of capacity notions

Now we turn to the first result of the paper. We show that all introduced capacities are the same in the sense that they give rise to the same capacity region. One can notice that once again the fundamental notion of teleportation finds its way to prove its usefulness.

Observation 1. *For multiparty quantum channel it holds* $\mathcal{Q}_g = \mathcal{Q}_m = \mathcal{Q}_e = \mathcal{Q}_s = \bar{\mathcal{Q}}_s$.

Remark: The problem of equivalence of different capacity notions in case of a multiple access channel was considered in [15]. Here, as in [14] and [16], we consider the most general scenario with k senders and m receivers. For bipartite case see [20].
Proof: ($\mathcal{Q}_e = \mathcal{Q}_s$) This equivalence holds for sources satisfying quantum asymptotic equipartition property. For proofs see [4] for bipartite and [16] multipartite case. For completeness of this paper we provide a revised multiparty proof in Appendix A.
($\bar{\mathcal{Q}}_s = \mathcal{Q}_s$) Generalization of the technique from Ref. [20] provides us with the equivalence. From a given reliable protocol we construct a new classically supported protocol which pure state fidelity equals average pure state fidelity of the original one. Uselessness of classical side channel finishes the proof. For details see the Appendix B.
($\bar{\mathcal{Q}}_s = \mathcal{Q}_m$) In the Appendix C we prove the generalization of the formula from Ref. [23]

connecting average fidelity with channels fidelity which with its local counterparts gives the desired. In particular, for global average fidelity we have

$$\bar{F}_s = \frac{1}{D_+}\left(DF_c + \sum_{j\in G}\frac{D}{d_j}F_c^{[G\setminus\{j\}]} \quad + \sum_{i,j\in G;i\neq j}\frac{D}{d_i d_j}F_c^{[G\setminus\{i,j\}]} + \right. \tag{2}$$

$$\left. \sum_{i,j,k\in G;i\neq j\neq k}\frac{D}{d_i d_j d_k}F_c^{[G\setminus\{i,j,k\}]} + \ldots + 1\right),$$

where $d_i = \dim \mathcal{H}_{A_i}, D = \Pi_{i\in G} d_i, D_+ = \Pi_{i\in G}(d_i+1)$. The formula implies that in the limit of large dimensions average fidelity tends to the channel fidelity, *i.e.* $\lim_{d_1,d_2,\ldots,d_k\to\infty}\bar{F}_s = F_c$. If average fidelity is close to one then all channel smaller group fidelities are also high. So maximal entanglement transmission and average subspace transmission are equivalent. For the details of the derivation of (2) see the Appendix C.
($\mathcal{Q}_m \subseteq \mathcal{Q}_g$) Consider a protocol for sending maximal entanglement. Encoding of ith sender results in some density matrix, which we can consider as a mixture of pure states, which, by convexity argument, means that for at least one component of the mixture we could achieve reliable transmission without the necessity of encoding. Consequently it implies existence of a protocol for generating entanglement with the rate at least as good as for transmission of it. Naturally we also have $\mathcal{Q}_m \subseteq \mathcal{Q}_g^{\rightarrow}$.
($\mathcal{Q}_g \subseteq \mathcal{Q}_s$) Generated entanglement can be used to perform teleportation with high fidelity. In this way we have $\mathcal{Q}_g \subseteq \mathcal{Q}_s^{\rightarrow}$ (by the same argument $\mathcal{Q}_g^{\rightarrow} \subseteq \mathcal{Q}_s^{\rightarrow}$ holds). The procedure uses forward communication, which, as stated previously, is useless *i.e.* $\mathcal{Q}_s^{\rightarrow} = \mathcal{Q}_s$. This inclusion is closely related to the problem of constructing a quantum error correction code from the distillation plus teleportation protocol [2]. ■
The above results immediately imply that $\mathcal{Q}_g^{\rightarrow} = \mathcal{Q}_g$. In a similar manner one shows that also the remaining scenarios do not gain any advantage acquiring free classical communication. For an interesting backward classical communication scenario see [24].

4. Capacity regions

4.1. Capacity theorem

We turn now to the second result of the paper. Namely, we give a multiletter characterization of the capacity region of the general km–user channel.

In case of $k = 1$ and $m \geq 2$ we obtain a broadcast channel capacity region; for $k \geq 2$ and $m = 1$ we get a multiple access channel, which capacity region was recently provided in Ref. [15] and was shown to be better than presented below for finite number n. When $k \geq 2$ and $m \geq 2$ these two scenarios coexist.

The Observation 2, which we state below, concerns zero-way capacity region equivalent to the one-way one. We prove the result in the entanglement generation scenario which according to Observation 1 is equivalent to other ones.

Observation 2 (Capacity region of a km–user channel) *Zero(one) – way capacity region $\mathcal{Q}(\Lambda)$ of a general km–user channel $\Lambda : \mathcal{B}(\mathcal{H}_{\mathbf{A'}}) \to \mathcal{B}(\mathcal{H}_{\mathbf{B}})$, ($\mathbf{A'} = \mathbf{A'_1 A'_2 ... A'_k}, \mathbf{B} = \mathbf{B_1 B_2 \ldots B_m}$) is given by the closure of $\bigcup_{n=1}^{\infty}\frac{1}{n}\widetilde{\mathcal{Q}}(\Lambda^{\otimes n})$, where $\widetilde{\mathcal{Q}}(\Lambda)$ is the union of km–tuple of nonnegative rates $\{\mathfrak{R}_i\}_{i\in G}$ satisfying $\mathfrak{R}_i < I_c(A_i > B_i)_{\varrho_{(AB)_i}}$, over all*

$\varrho_{\mathbf{AB}} = \left(\mathcal{I}^{\mathbf{A}} \otimes \Lambda_{\mathbf{A'}\to\mathbf{B}}\right)\left(\bigotimes_{\omega\in\mathcal{K}} \Psi_{(\mathbf{AA'})_\omega}\right)$ *which* $\varrho_{(AB)_i}$ *arise from by tracing out all the systems besides i–th one.*

Proof: (achievability) Alicias produce $\left[\varrho^{(n)}_{\mathbf{AB}}\right]^{\otimes\tilde{n}} \equiv \left[\left(\mathcal{I}^{\mathbf{A}} \otimes \Lambda^{\otimes n}_{\mathbf{A'}\to\mathbf{B}}\right)\left(\bigotimes_{\omega\in\mathcal{K}} \Psi^{(n)}_{(\mathbf{AA'})_\omega}\right)\right]^{\otimes\tilde{n}}$ and perform with Bobbys one–way hashing protocol of Devetak and Winter [7] on $\left(\varrho^{(n)}_{(AB)_j}\right)^{\otimes\tilde{n}}$ which achieves asymptotically entanglement generation rates $\frac{1}{n}I_c(A_j > B_j)_{\varrho^{(n)}_{(AB)_j}}$. Since forward communication is useless the rates are achievable in zero–way communication.

Before we proceed we recall a useful lemma (see [11])

Lemma;- For states $\varrho^{\mathcal{AB}}$ *and* $\sigma^{\mathcal{AB}}$*, of the same d dimensions, with fidelity* $F(\varrho^{\mathcal{AB}}, \sigma^{\mathcal{AB}}) \equiv (\mathrm{tr}\,|\sqrt{\varrho^{\mathcal{AB}}}\sqrt{\sigma^{\mathcal{AB}}}|)^2 := 1 - f$ *we have* $|I_c(\mathcal{A} > \mathcal{B})_{\varrho^{\mathcal{AB}}} - I_c(\mathcal{A} > \mathcal{B})_{\sigma^{\mathcal{AB}}}| \leq 4\sqrt{f}\log d + 2$.

(converse) Consider entanglement generation protocol achieving rates $\mathfrak{R}^g_i = \limsup_{n\to\infty} \mathcal{R}^{g(n)}_i$, where $\mathcal{R}^{g(n)}_i := \log d^{(n)}_j/n$ We have $F^{(i)}_g\left(\bigotimes_{j\in G} \Psi^{(n)}_{(\mathbf{AA'})_\mathbf{j}}, \bigotimes_{l\in\mathcal{M}} \mathcal{D}^{(n)}_l \circ \Lambda^{\otimes n}\right) = 1 - \eta_n$ with $\eta_n \to 0$ for $n \to \infty$. Now taking in the Lemma $\varrho^{\mathcal{AB}}$ as $\mathrm{tr}_{\mathbf{AB}\setminus(AB)_i}\left(\mathcal{D}^{(n)} \circ \Lambda^{\otimes n}\left(\bigotimes_{j\in\mathcal{K}} \Psi^{(n)}_{(\mathbf{AA'})_\mathbf{j}}\right)\right) \equiv \tilde{\mathcal{D}}^{(n)}_i(\varrho_{(AB)_i})$ and $\sigma^{\mathcal{AB}} = \Phi^{(+)}_{d^{(n)}_i(AB)_i}$ we have the following justified by the data processing inequality and the Lemma: $I_c(A_i > B_i)_{\varrho_{(AB)^{(n)}_i}} \geq I_c(A_i > B_i)_{\tilde{\mathcal{D}}^{(n)}_i(\varrho^{(n)}_{(AB)_i})} \geq n\mathfrak{R}^{g(n)}_i - 2 - 8\sqrt{\eta}n\mathfrak{R}^{g(n)}_i \geq n(\mathfrak{R}^{g(n)}_i - \delta_\eta)$ with $\delta_\eta \to 0$ when $n \to \infty$. This concludes the proof since the claimed set is closed. ■

One can easily verify that the region does not require convexification (cf. [15]).

4.2. Generalization to the two-way quantum capacity regions

In a sense the above regions were derived by extended reasoning of [9] in that it utilizes (apart from data processing inequality) two elements: hashing inequality for entanglement distillation [7] and the fact that forward communication does not improve quantum capacity ([16], [4]). So it is natural to ask about possibility of extending the present results to the case of two-way communication as it was in [9]. The answer is positive. All the reasoning leading to theorems above uses either zero-way (encoding, decoding) or one-way protocols (teleportation). As it was in bipartite case one can follow any protocol achieving some fixed coherent information rates by one-way protocol involving entanglement distillation and teleportation. The above leads to the following simple conclusions: *Observation 1 is valid for all capacities if we involve two-way encoding-decoding procedure. Also capacity regions provided in Observation 2 are true if only in a place of the state we put arbitrary state that can be produced with help of a quantum channel* $\Lambda^{\otimes n}$ *assisted by two-way LOCCs. Finally, the multiple access channel's capacity region provided in [22, 15] can be extended to two-way case in such a manner.*

5. Summary and Discussion

We have rigorously defined entanglement transmission, subspace transmission and entanglement generations in case of multiparty quantum channels, systematized known facts about equivalences of capacity notions, and shown truthfulness of the above in case of any multiuser communication scenario. Using this fact with the aid of recently proved

uselessness of unlimited forward classical communication we provided capacity regions for km-user channels, which special cases are the broadcast, multiple access, and k–user channels. It seems that further improvements of the region providing better approximations for finite n not involving some assumptions about the specific channel may be difficult. It would be also desirable to find single-letter characterizations for classes of channels. However, at this point this remains an open question. Finally we have pointed out elementary generalization of the results to the case of two-way capacity regions. In future it would be interesting to study the gap between the case of zero-way (one-way) case and the case of two-way supported quantum channel in a general km-user scenario.

After having had completed the main part of this work (quant-ph/0603112) we have become aware of the result of the Ref. [26] (quant-ph/0603098) where broadcast channels were considered.

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7. Appendices

Appendix A

For completeness of the paper we recall, with some details refined, the proof of equivalence between entanglement and subspace transmission [4, 16]. First we prove that entanglement transmission implies subspace transmission [28].

We assume that $F_e(\bigotimes_{i\in G}\varrho_{A_i},\Lambda)\geq 1-\eta$, where ϱ_{A_i} are the normalized density matrices of the transmitted QAEP sources $\mathfrak{S}_i$ projected onto their typical subspaces with $\dim\mathrm{supp}(\varrho_{A_i})=K_i$ (this projection does not decrease substantially the fidelity; [4]). Consider the following strategy. We find a vector $|\varphi^{(1)}_{A_1}\rangle\in\mathrm{supp}(\varrho_{A_1})$ that minimizes fidelity of the state $\varphi^{(1)}_{A_1}\otimes\left(\bigotimes_{i\in G\setminus\{1\}}\varrho^{(n)}_{A_i}\right)$; we will refer to this fidelity as to $F_{e,s}$ as this is of mixed type. We construct an operator $\rho^{(1)}_{A_1}=\varrho_{A_1}-q^1_1\varphi^{(1)}_{A_1}$ taking q^1_1 as large as we can still protecting positive semi-definiteness of the operator. We can proceed with the same strategy until we reach a zero operator. By construction in step $k+1$ we obtain an operator of dimension one less than in step k (we have removed one dimension from the support). What is more we get that $\{q^1_m,\varphi^{(m)}_{A_1}\}$ constitutes a pure state ensemble for ϱ_{A_1}, *i.e.* $\varrho_{A_1}=\sum_{m=1}^{K_1}q^1_m\varphi^{(m)}_{A_1}$. Let us assume that using this strategy we removed d_1 dimensions from the K_1-dimensional support obtaining a subspace $\mathcal{H}_{D_1}$, from which for all states we have $F_{e,s}\geq 1-\gamma_1$. If we further denote $\alpha_1=\sum_{m=1}^{d_1}q^1_m$ and by $\varrho^{d_i}_{A_i}$ normalized density matrix with d_1 dimensions removed we can rewrite global entanglement condition as $F_e\left(\left(\sum_{m=1}^{d_1}q^1_m\varphi^{(m)}_{A_1}+(1-\alpha_1)\varrho^{d_1}_{A_1}\right)\otimes\bigotimes_{i\in G\setminus\{1\}}\varrho_{A_i},\Lambda\right)\geq 1-\eta$. By convexity of entanglement fidelity in the input density operator, *i.e.* $F_e(\sum_i p_i\rho_i,\Lambda)\leq\sum_i p_iF_e(\rho_i,\Lambda)$, we get $1-\eta\leq(1-\gamma_1)\alpha_1+(1-\alpha_1)$, which gives $\gamma_1\leq\eta/\alpha_1$. Repeating the above procedure for the rest $i\in G\setminus\{1\}$ we obtain $F_s(\bigotimes_{i\in G}\mathcal{H}_{D_i},\Lambda)\geq 1-\gamma_{|G|}$, where $\gamma_{|G|}\leq\eta/\Pi_{i\in G}\alpha_i$. We thus obtained a factor by which the entanglement fidelity is

decreased when we consider transmission of $\mathcal{H}_{D_i}$. Next we get the bounds for the dimensions of these subspaces. For all i and m we obviously have $q^i_m \leq \lambda_{max}(\varrho_{A_i})$, which due to the fact that ϱ_{A_i} are normalized density operators restricted to the typical subspaces can be further bounded from above by $2^{-n(\Re_S(\mathfrak{S}_i)-\epsilon_i)}/(1-\delta_i)$ with the denominator being the probability of the projection onto the typical subspace. All these combined gives $D_i \geq (1-\alpha_i)|T^{i(n)}_\epsilon|, |T^{i(n)}_\epsilon| = (1-\delta_i)2^{n(\Re_S(\mathfrak{S}_i)-\epsilon)}$. This implies, with the initial assumption of having had arbitrarily taken ϱ_i into account, that the same region for entanglement transmission can also be achieved for subspace transmission.

Now let us move to another direction of implication. Using the technique of Ref. [4, 25] we will show that if the product Hilbert space is reliably sent through the channel then product density matrix supported on the subspace of it can also be sent with high fidelity. Suppose that $F_s(\bigotimes_{i\in G}\mathcal{H}_{A_i},\Lambda) \geq 1-\eta$, *i.e.* $F_s(\bigotimes_{i\in G}\varphi_{A_i},\Lambda) \geq 1-\eta$ for all $\varphi_{A_i} \in \mathcal{H}_{A_i}$, $i \in G$. Writing the first state as a superposition of basis states *i.e.* $\varphi_{A_1} = \sum_k \sqrt{\lambda^1_k} \mathrm{e}^{\mathrm{i}\phi^1_k}|k_1\rangle$ and putting it to the above condition followed by averaging over phases, which does not decrease fidelity, one obtains

$$\bar{F}_s = \sum_{kl} \lambda^1_k \lambda^1_l \langle k_1| \left(\bigotimes_{i\in G\setminus\{1\}} \langle\varphi_{A_i}|\right) \Lambda \left(|k_1\rangle\langle l_1| \otimes \bigotimes_{i\in G\setminus\{1\}} \varphi_{A_i}\right) |l_1\rangle \left(\bigotimes_{i\in G\setminus\{1\}} |\varphi_{A_i}\rangle\right) +$$

$$\sum_{km,k\neq m} \lambda^1_k \lambda^1_m \langle m_1| \left(\bigotimes_{i\in G\setminus\{1\}} \langle\varphi_{A_i}|\right) \Lambda \left(|k_1\rangle\langle k_1| \otimes \bigotimes_{i\in G\setminus\{1\}} \varphi_{A_i}\right) |m_1\rangle \left(\bigotimes_{i\in G\setminus\{1\}} |\varphi_{A_i}\rangle\right).$$

Direct calculation shows that the first term is the fidelity of the state $\varrho_{A_1} \otimes \left(\bigotimes_{i\in G\setminus\{1\}}\varphi_{A_i}\right)$, $\varrho_{A_1} = \sum_k \lambda^1_k |k_1\rangle\langle k_1|$, sent through the channel. By the same arguments as in Ref. [4] one can show that $F_{e,s}\left(\varrho_{A_1}\otimes\left(\bigotimes_{i\in G\setminus\{1\}}\varphi_{A_i}\right),\Lambda\right) \geq 1-\frac{3}{2}\eta$. We follow with the same strategy of averaging which results in a bound for entanglement fidelity as follows $F_e\left(\bigotimes_{i\in G}\varrho_{A_i},\Lambda\right) \geq 1-(\frac{3}{2})^{|G|}\eta$. To argue that the same capacity region for subspace transmission is also achievable for entanglement transmission we take uniform density matrices on the transmitted spaces.

Appendix B

We follow modified strategy of Ref. [20] to prove the desired equivalence. We supplement the protocol with a specially constructed classical forward channel so that the new channel is as follows

$$\mathcal{N}^{\rightarrow}(\cdot) = \sum_{\vec{n}} \frac{1}{\Pi_{i\in G}N_i} \bigotimes_{i\in G} U^\dagger_{n_i} \mathcal{N}\left(\bigotimes_{i\in G} U_{n_i}(\cdot)\bigotimes_{i\in G} U^\dagger_{n_i}\right) \bigotimes_{i\in G} U_{n_i}. \tag{B-1}$$

The vector $\vec{n} = (n_1, n_2, ..., n_{|G|})$ represents a classical message sent to receivers, n_i are taken from N_i elements sets. At this moment we refrain from specifying the sets of unitary U. Consider now pure state fidelity in our scenario which, assuming that $\mathcal{N}(\cdot) = \sum_j A_j(\cdot)A^\dagger_j$ and $N = \Pi^{|G|}_{i=1}N_i$, yields

$$F_s\left(\bigotimes_{i\in G}\psi_{A_i},\mathcal{N}^{\rightarrow}\right) = \frac{1}{N}\sum_j\sum_{\vec{n}}\left(\bigotimes_{i\in G}\langle\psi_{A_i}|U^\dagger_{n_i}\right)A_j\left(\bigotimes_{i\in G}U_{n_i}\psi_{A_i}U^\dagger_{n_i}\right)A^\dagger_j\left(\bigotimes_{i\in G}U_{n_i}|\psi_{A_i}\rangle\right)$$

Now we ask whether we can replace the sums with integrals and how should be the sets of U be chosen if the answer is positive. We discuss these questions in what follows.

Define an operation

$$\mathcal{N}^{(1)}_{\vec{n}\setminus n_1}(\cdot) = \sum_j A^{(1)j}_{\vec{n}\setminus n_1}(\cdot)(A^{(1)j}_{\vec{n}\setminus n_1})^\dagger, \tag{B-2}$$

where Kraus operators are defined by the partial inner product

$$A^{(1)j}_{\vec{n}\setminus n_1} = \left(\bigotimes_{i\in G\setminus\{1\}} \langle\psi_{A_i}|U^\dagger_{n_i}\right) A_j \left(\bigotimes_{i\in G\setminus\{1\}} U_{n_i}|\psi_{A_i}\rangle\right). \tag{B-3}$$

Eq. (B-2) then takes the form

$$F_s = \sum_{\vec{n}\setminus n_1} \frac{N_1}{N}\langle\psi_{A_1}|\left(\frac{1}{N_1}\sum_{n_1} U^\dagger_{n_1}\mathcal{N}^{(1)}_{\vec{n}\setminus n_1}(U_{n_1}\psi_{A_1}U^\dagger_{n_1})U_{n_1}\right)|\psi_{A_1}\rangle. \tag{B-4}$$

From the theory of unitary 2–designs [27] we know that in cases when we deal with $U(2^N)$ we have an equivalence $\frac{1}{K}\sum_k U_k^\dagger\mathcal{N}(U_k\varrho U_k^\dagger)U_k = \int \mathrm{d}U U^\dagger\mathcal{N}(U\varrho U^\dagger)U$ with suitable chosen $\{U_k\}$, which were shown to be the Clifford group $\mathcal{C}_N$. We can directly use this fact since here we deal with spaces of the proper dimensions. This follows from the possibility of bounding the dimensions of the transmitted spaces in the following manner $2^{l_n} \leq d_n \leq 2^{l_n+1}$ with $l_n \to \infty$ when $n \to \infty$ and restricting ourselves to the spaces of dimension from the LHS of the first inequality which leaves rates and fidelities unchanged. This turns Eq. (B-4) into

$$F_s = \sum_{\vec{n}\setminus n_1} \frac{N_1}{N}\langle\psi_{A_1}|\left(\int \mathrm{d}U_{n_1} U^\dagger_{n_1}\mathcal{N}^{(1)}_{\vec{n}\setminus n_1}(U_{n_1}\psi_{A_1}U^\dagger_{n_1})U_{n_1}\right)|\psi_{A_1}\rangle. \tag{B-5}$$

Now taking back the step (B-2) and applying analogous procedure to the remaining $|G|-1$ states we obtain

$$F_s = \int \Pi_{i\in G}\mathrm{d}U_{n_i}\bigotimes_{i\in G}\langle\psi_{A_i}|U^\dagger_{n_i}\mathcal{N}\left(\bigotimes_{i\in G} U_{n_i}\psi_{A_i}U^\dagger_{n_i}\right)\bigotimes_{i\in G} U_{n_i}|\psi_{A_i}\rangle, \tag{B-6}$$

which is just the average pure state fidelity $\overline{F}_s$. By the uselessness of classical channel we conclude the equivalence of capacities.

Appendix C

Here we prove Eq. (2). In what follows sub- and superscripts denoted with $\mathcal{H}$ will indicate spaces on which operators act, channel Λ has Kraus operators $\{A_K\}_K$. Using the approach from Ref. [20] we arrive at:

$$\bar{F}_s(\Lambda) = \bar{D}\sum_K \mathrm{tr}\, A_K^{\mathcal{H}_{1,2,\ldots,|G|}\dagger} \otimes A_K^{\mathcal{H}_{|G|+1,|G|+2,\ldots,2|G|}} \bigotimes_{i\in G}(\mathbb{I}_{\mathcal{H}_{i,|G|+i}} + \mathbb{V}_{\mathcal{H}_{i,|G|+i}}),$$

where $\bar{D} = \Pi_{i\in G} d_i^{-1}(d_i+1)^{-1}$ which can be rewritten as

$$\begin{aligned}
\bar{F}_s(\Lambda) = \bar{D}\sum_K \mathrm{tr}\, A_K^{\mathcal{H}_{1,2,\ldots,|G|}\dagger} \otimes A_K^{\mathcal{H}_{|G|+1,|G|+2,\ldots,2|G|}} \Big[\mathbb{I}_{\mathcal{H}_{1,2,\ldots,|G|,|G|+1,\ldots,2|G|}} + \\
\sum_\pi \big(\frac{1}{1!(|G|-1)!}\mathbb{V}_{\mathcal{H}_{1,|G|+1}} \otimes \mathbb{I}_{\mathcal{H}_{G^2\setminus\{1,|G|+1\}}} + \frac{1}{2!(|G|-2)!}\times \\
\mathbb{V}_{\mathcal{H}_{1,|G|+1}} \otimes \mathbb{V}_{\mathcal{H}_{2,|G|+2}} \otimes \mathbb{I}_{\mathcal{H}_{G^2\setminus\{1,2,|G|+1,|G|+2\}}} + \ldots\big) + \mathbb{V}_{\mathcal{H}_{1,2,\ldots,|G|,|G|+1,|G|+2,\ldots,2|G|}}\Big],
\end{aligned} \tag{C-1}$$

where the permutation π permutes Hilbert spaces $\mathcal{H}_{i,|G|+i} = \mathbb{C}^{d_i}\otimes\mathbb{C}^{d_i}$ and $G^2 = \{1,2,\ldots,|G|,|G|+1,\ldots,2|G|\}$. In general $\mathcal{H}_{klm\cdots} \equiv \mathcal{H}_k\otimes\mathcal{H}_l\otimes\mathcal{H}_m\otimes\cdots$ and $\mathbb{V}_{\mathcal{H}_{\vec{a},\vec{b}}}|\phi\rangle|\psi\rangle = |\psi\rangle|\phi\rangle$, $|\phi\rangle\in\mathcal{H}_{\vec{a}}$, $|\psi\rangle\in\mathcal{H}_{\vec{b}}$, $|\vec{a}| = |\vec{b}|$. Now we will associate all the terms in the above with the proper channel fidelities.

Let us start with a calculation of global channel fidelity. We have

$$\begin{aligned}
F_c(\Lambda) = \quad & \sum_K \mathrm{tr}\left[\mathbb{I}_{\mathcal{H}_{1,2,\ldots,|G|}} \otimes A_K^{\mathcal{H}_{|G|+1,|G|+2,\ldots,2|G|}}\left(\bigotimes_{i\in G} P_+^{\mathcal{H}_{i,i+|G|}}\right)\times\right. \\
&\left.\times\, \mathbb{I}_{\mathcal{H}_{1,2,\ldots,|G|}} \otimes A_K^{\mathcal{H}_{|G|+1,|G|+2,\ldots,2|G|}\dagger}\left(\bigotimes_{i\in G} P_+^{\mathcal{H}_{i,i+|G|}}\right)\right],
\end{aligned}$$

where $P_+ = |\Phi^{(+)}\rangle\langle\Phi^{(+)}|$ is a maximally entangled state projector acting on $\mathbb{C}^d$. Now, after having used the following properties: $\mathrm{tr}\, A_{12}^{\Gamma_1}B_{12}^{\Gamma_1} = \mathrm{tr}\, A_{12}B_{12}$, $\mathrm{tr}\,(\mathbb{I}_1\otimes A_2\varrho_{12}\mathbb{I}_1\otimes B_2)^{\Gamma_1} = \mathrm{tr}\,\mathbb{I}_1\otimes A_2\varrho_{12}^{\Gamma_1}\mathbb{I}_1\otimes B_2$, and $dP_+^\Gamma = \mathbb{V}$ in the order they are quoted here, we obtain

$$\begin{aligned}
F_c(\Lambda) = \quad & \frac{1}{\Pi_{i\in G}d_i^2}\sum_K \mathrm{tr}\left[\mathbb{I}_{\mathcal{H}_{1,2,\ldots,|G|}} \otimes A_K^{\mathcal{H}_{|G|+1,|G|+2,\ldots,2|G|}}\left(\bigotimes_{i\in G}\mathbb{V}_{\mathcal{H}_{i,i+|G|}}\right)\times\right. \\
&\left.\times\, \mathbb{I}_{\mathcal{H}_{1,2,\ldots,|G|}} \otimes A_K^{\mathcal{H}_{|G|+1,|G|+2,\ldots,2|G|}\dagger}\left(\bigotimes_{i\in G}\mathbb{V}_{\mathcal{H}_{i,i+|G|}}\right)\right],
\end{aligned}$$

which can further be rewritten as

$$F_c(\Lambda) = \frac{1}{\Pi_{i\in G}d_i^2}\sum_K \mathrm{tr}\, A_K^{\mathcal{H}_{1,2,\ldots,|G|}\dagger} \otimes A_K^{\mathcal{H}_{|G|+1,|G|+2,\ldots,2|G|}},$$

which, up to a constant factor, is the first term in the considered sum.

Let us now move to group fidelities. We will describe our procedure in details for group fidelities of order $|G|-1$ as the method easily generalizes. These will be denoted $F^{[G\setminus\{k\}]}$, where k is the enumeration of connection which is traced out. We give a method of calculation of $F^{[G\setminus\{k\}]}$ for all $k\in G$ involving only one direct calculation which we provide below. We have

$$\begin{aligned}
F_c^{[G\setminus\{k\}]}(\Lambda) = \mathrm{tr}\left[\left(\mathrm{tr}_{\,k,k+|G|}\mathcal{I}_{\mathcal{H}_{1,2,\ldots,|G|}} \otimes \Lambda_{\mathcal{H}_{|G|+1,|G|+2,\ldots,2|G|}}\left(\bigotimes_{i\in G}P_+^{\mathcal{H}_{i,i+|G|}}\right)\right)\times\right. \\
\left.\times\left(\bigotimes_{i\in G\setminus\{k\}}P_+^{\mathcal{H}_{i,i+|G|}}\right)\right] = \mathrm{tr}\left[\left(\mathcal{I}_{\mathcal{H}_{1,2,\ldots,|G|}} \otimes \Lambda_{\mathcal{H}_{|G|+1,|G|+2,\ldots,2|G|}}\left(\bigotimes_{i\in G}P_+^{\mathcal{H}_{i,i+|G|}}\right)\right)\times\right.
\end{aligned}$$

$$\times \left(\bigotimes_{i\in G\setminus\{k\}} P_+^{\mathcal{H}_{i,i+|G|}} \otimes \mathbb{I}_{\mathcal{H}_{k,k+|G|}} \right) \Bigg],$$

where we have used the property tr $A\varrho_1 =$ tr $A^1 \otimes \mathbb{I}^2 \varrho_{12}$. Decomposition of Λ into its Kraus components, application of the previously used properties allow us to write

$$F_c^{[G\setminus\{k\}]}(\Lambda) = \frac{1}{\Pi_{i\in G\setminus\{k\}} d_i^2 d_k} \sum_K \mathrm{tr} \Bigg[\mathbb{I}_{\mathcal{H}_{1,2,\ldots,|G|}} \otimes A_K^{\mathcal{H}_{|G|+1,|G|+2,\ldots,2|G|}} \left(\bigotimes_{i\in G} \mathbb{V}_{\mathcal{H}_{i,i+|G|}} \right)$$
$$\times \quad \mathbb{I}_{\mathcal{H}_{1,2,\ldots,|G|}} \otimes A_K^{\mathcal{H}_{|G|+1,|G|+2,\ldots,2|G|}\dagger} \left(\bigotimes_{i\in G\setminus\{k\}} \mathbb{V}_{\mathcal{H}_{i,i+|G|}} \otimes \mathbb{I}_{\mathcal{H}_{k,k+|G|}} \right) \Bigg].$$

Finally, inserting identity divided into swaps before the last term under the trace gives

$$F_c^{[G\setminus\{k\}]}(\Lambda) = \frac{\sum_K \mathrm{tr}\ A_K^{\mathcal{H}_{1,2,\ldots,|G|}\dagger} \otimes A_K^{\mathcal{H}_{|G|+1,|G|+2,\ldots,2|G|}} \left(\mathbb{I}_{\mathcal{H}_{G^2,\setminus\{k,k+|G|\}}} \otimes \mathbb{V}_{\mathcal{H}_{k,k+|G|}} \right)}{\Pi_{i\in G\setminus\{k\}} d_i^2 d_k}.$$

Consequently, these group fidelities give rise to the terms with only one swap in (C-1). We apply previously described procedure to the remaining terms besides the last one which is easily found to be equal to $\Pi_{i\in G}(d_i+1)^{-1}$. All above results give us Eq. (2).

Within the same method group fidelities analogs can be obtained.

References

[1] B. Schumacher, Phys. Rev. A 51 (1995) 2738; B. Schumacher, Phys. Rev. A 54 (1996) 2614.
[2] C.H. Bennett, D.P. DiVincenzo, J.A. Smolin, W.K. Wootters, Phys. Rev. A. 54 (1996) 3824.
[3] H. Barnum, M.A. Nielsen and B. Schumacher, Phys. Rev. A 57 (1998) 4153.
[4] H. Barnum, E. Knill, M.A. Nielsen, IEEE Trans. Inf. Th. 46 (2000) 19.
[5] C.H. Bennett, D.P. DiVincenzo and J.A. Smolin, Phys. Rev. Lett. 78 (1997) 3217.
[6] H.Barnum, J. A. Smolin, B. Terhal, Phys. Rev. A 58 (1998) 3496.
[7] I. Devetak, A. Winter, Phys. Rev. Lett., 93 (2004) 080501; Proc. R. Soc. Lond. A 461 (2005) 207.
[8] S. Lloyd, Phys. Rev. A. 55 (1997) 1613.
[9] M. Horodecki, P. Horodecki, R. Horodecki, Phys. Rev. Lett. 85 (2000) 433.
[10] P. W. Shor, MSRI Workshop on Quantum Computation lecture notes 2002, www.msri.org/publications/ln/msri/2002/quantumcrypto/shor/1/
[11] I. Devetak, IEEE Trans. Inf. Th. IT-55 (2005) 44.
[12] C. H. Bennett, I. Devetak, P. W. Shor, J. A. Smolin, Phys. Rev. Lett. 96 (2006) 150502.
[13] G. Smith and J. Smolin, Science 321 (2008) 1812.
[14] W. Dür, J. I. Cirac, and P. Horodecki, Phys. Rev. Lett. 93 (2004) 020503.
[15] J. Yard, I. Devetak, P. Hayden, IEEE Trans. Inf. Th. 54 (2008) 3091.
[16] M. Demianowicz, P. Horodecki, Phys. Rev. A 74 (2006) 042336.
[17] M. B. Hastings, Nature Physics 5 (2009) , 255.
[18] L. Czekaj and P. Horodecki, Phys. Rev. Lett. 102 (2009) 110505.
[19] K. Li, A. Winter, X. B. Zou, and G. C. Guo, Phys. Rev. Lett. 103 (2009), 120501.
[20] D. Kretschmann, R. Werner, New J. Phys. 6 (2004) 26.
[21] This approach (see [9, 20]) is somewhat more general than requiring $\mathcal{H}^{(n)} = \mathcal{H}^{\otimes n}$.
[22] M. Horodecki, J. Oppenheim, A. Winter, Nature 436 (2005) 673; M. Horodecki, J. Oppenheim, A. Winter, Comm. Math. Phys. 269 (2007) 107 .
[23] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A 60 (1999) 1888 .
[24] A. W. Leung, Phys. Rev. A **77**, 012322 (2008); the quantity E_B defined there and called entanglement capacity with classical feedback is just $\mathcal{Q}_g^{\leftarrow}$ in our notation.

[25] E. Knill and R. Laflamme, Phys. Rev. A **55**, 900 (1997).
[26] J. Yard, P. Hayden, I. Devetak, Report No. quant-ph/0603098.
[27] C. Dankert, MSc Thesis, University of Waterloo, 2005, Report No. quant-ph/0512217; C. Dankert, R. Cleve, J. Emerson, and E. Livine, Report No. quant-ph/0606161, D. Gross, K. Audenaert, J. Eisert, Report No. quant-ph/0611002.
[28] To simplify notation we omit superscript (n) in Appendices.